Differential Geometry IV: General Relativity

2.1 On Minkowski spacetime $(\mathbb{R}^{n+1},\eta)$, let $(x^0,\dots,x^n)$ be the standard Cartesian coordinate system. Compute the induced metric on the submanifolds

$$S^{(n,1)}_{-1}=\Big\{-(x^0)^2+\sum_{i=1}^{n}(x^i)^2=-1\Big\}$$

and

$$S^{(n,1)}_{+1}=\Big\{-(x^0)^2+\sum_{i=1}^{n}(x^i)^2=+1\Big\}$$

(the latter is known as *de-Sitter* spacetime).

2.2 On Minkowski spacetime $(\mathbb{R}^{n+1},\eta)$, let $p,q\in\mathbb{R}^{n+1}$ be two points such that $q\in I^+(p)$. Let also $\gamma_0:[0,1]\to\mathbb{R}^{n+1}$ be the straight line segment connecting p to q (i.e. $\gamma_0(0)=p$, $\gamma_0(1)=q$ and $\ddot\gamma_0=0$) and $\gamma:[0,1]\to\mathbb{R}^{n+1}$ be any other *causal* curve (i.e. with $\dot\gamma$ either timelike or null) such that $\gamma(0)=p$ and $\gamma(1)=q$. Show that the corresponding lengths of the curves satisfy

$$\ell(\gamma_0)\geqslant\ell(\gamma).$$

This is a manifestation of the *twin paradox* in special relativity.
Hint: Approximate γ by a polygonal causal curve and, using the inverse triangle inequality for causal vectors, show that the line segment connecting p and q has greater or equal length to a broken line segment connecting the same points.

2.3 Let (M,g) be a smooth Lorentzian *surface* (i.e. 2-dimensional manifold).

(a) Show that for any $p\in\mathcal{M}$, there exists an open neighborhood $\mathcal{U}$ of p and a local system of coordinates (u,v) on $\mathcal{U}$ such that

$$g=\Omega(u,v)dudv,$$

where $\Omega\in C^\infty(\mathcal{U})$ does not vanish in $\mathcal{U}$ (such a coordinate system is called a *characteristic* or *double null* system).

(b) Deduce that every smooth Lorentzian surface is locally conformally equivalent to an open subset of the Minkowski space $(\mathbb{R}^{1+1},\eta)$ (recall that a similar fact also holds for *Riemannian* surfaces; in that case, a coordinate system exhibiting this equivalence is called *isothermal*).

***2.4** In this exercise, we will show that there are topological obstructions to a manifold admitting a Lorentzian metric; **not** every smooth manifold admits one. To this end, let us adopt the following definition: For any Lorentzian inner product space (V,m), we will call any 2-element set of the form $\{u,-u\}$ (where $u\in V\setminus 0$) a *line seed.* A line seed $X=\{u,-u\}$ will be called *causal* if $u\in V$ is a causal vector. We will also define the *trivial* line seed to be the pair $\{0,-0\}$.

Given two causal line seeds $X+\{u,-u\}$ and $Y=\{v,-v\}$, then exactly one of the vectors $+v$ and $-v$ belongs to the same timecone as u. We will define the sum $X+Y$ as the seed $\{u+v,-u-v\}$ if u,v belong to the same time cone and as $\{u-v,-u+v\}$ otherwise. We will extend this definition to include the trivial line seed.

(a) Verify that, with the addition operator defined above, $X_1 + X_2$ is a causal line seed if X_1, X_2 are causal line seeds or if one of them is causal and the other is the trivial line seed.

(b) Let $(\mathcal{M}, g)$ be a smooth Lorentzian manifold and let $p \in \mathcal{M}$. Show that there exists an open neighborhood $\mathcal{U}$ of p and a smooth causal vector field $U \in \Gamma(\mathcal{U})$.

(c) A smooth *line field seed* on $\mathcal{M}$ will be an assignment of a line seed $X_p = \{U_p, -U_p\}$ in $T_p\mathcal{M}$ for each $p \in \mathcal{M}$ such that, for any $q \in \mathcal{M}$, there exists an open neighborhood $\mathcal{V}$ of q and a smooth vector field $Y \in \Gamma(\mathcal{V})$ such that $Y(p) \in X_p$ for all $p \in \mathcal{V}$.[1] Show that $\mathcal{M}$ as above admits a smooth *causal* line field seed.
Hint: For this part, it might be helpful to use the fact that any smooth manifold admits a ***partition of unity****: For any open covering $\{\mathcal{U}_a\}_a$ of $\mathcal{M}$, there exists a family $\{\chi_\beta\}_\beta$ of smooth functions $\chi_\beta : \mathcal{M} \to [0, +\infty)$ satisfying the following properties:*

- *Each χ_β is compactly supported, and its support is contained in one of the open sets $\mathcal{U}_a$.*
- *For each χ_β, supp(χ_β) intersects only finitely many of the supports of χ_γ, $\gamma \neq \beta$.*
- *For any $p \in \mathcal{M}$, $\sum_\beta \chi_\beta(p) = 1$.*

You can then use part 2.4.b to construct a smooth causal line seed field in a neighborhood of every point in $\mathcal{M}$, and then use an appropriate partition of unity to "glue" these constructions together, utilising the notion of the sum of two causal line seeds from part 2.4.a.

(d) Deduce that the tangent bundle $T\mathcal{M}$ of $\mathcal{M}$ admits a smooth line subbundle. Can the sphere $\mathbb{S}^2$ admit a Lorentzian metric?
Hint: Use the fact that, for a compact manifold $\mathcal{M}$, if the tangent bundle admits a line subbundle then the Euler characteristic $\chi(\mathcal{M})$ of $\mathcal{M}$ vanishes.

Bonus exercise (hard): Can you construct a Lorentzian metric on $\mathbb{S}^3$?

[1] Note that, with this definition, the tangent vector U_p need not even be continuous in p; we only require that there is (locally at least) a choice between U_p and $-U_p$ at every point p that results in a smooth vector field.